Chapter 4: Measures of Image Quality

Slide set of 184 slides based on the chapter authored by A.D.A. Maidment, PhD of the IAEA publication (ISBN 978-92-0-131010-1):

Diagnosis Radiology Physics: A Handbook for Teachers and Students

Objective:
To familiarise the student with methods of quantifying image quality.

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4.1 INTRODUCTION

A medical image is a **Pictorial Representation** of a measurement of an object or function of the body.

Many different ways exist to acquire medical image data.
4.1 INTRODUCTION

Knowledge of image quality allows for comparison of imaging system designs:

- **Within** a modality, and
- **Across** Different imaging modalities

This information can be acquired in 1-3 spatial dimensions

It can be **Static** or **Dynamic**, meaning that it can be measured also as a function of time
Fundamental properties associated with these data:

- No image can **Exactly** represent the object or function; at best, one has a measurement with an associated error equal to the difference between the true object and the measured image.
- No two images will be **Identical**, even if acquired with the same imaging system of the same anatomic region variability generally referred to as **Noise**.
There are many **Different** ways to acquire medical image data.

Regardless of the method, one must be able to judge the **Fidelity** of the image in an attempt to answer the question:

**How Accurately Does the Image Portray the Body or the Bodily Function?**
This judgment falls under the rubric of

Image Quality

Methods of **Quantifying** image quality are described in this chapter
Knowledge of image quality allows \textit{Comparison} of:

- Various imaging system designs for a given modality and
- Information contained in images acquired by different imaging modalities

The impact of image quality on an imaging task, such as \textbf{Detection} of a lesion in an organ, can also be determined.
Various imaging tasks require **Differing Levels** of image quality.

An image may be of sufficient quality for **One** task, but inadequate for **Another** task.
4.1 INTRODUCTION

The metrics introduced here are much used in the following chapters in this Handbook as the

- Design
- Performance, and
- Quality Control

of different imaging systems are discussed

First, however, one needs to learn the meaning of:

High Image Quality
4.2 IMAGE THEORY FUNDAMENTALS

4.2.1 Linear Systems Theory

In all imaging systems the output, $g$, is a function of the input, $f$.

The function, $H$, is usually called the Transfer Function or System Response Function.

For a continuous 2D imaging system, this relationship can be written as:

$$g(x, y) = H\{f(x, y)\}$$
The simple concept implies that we can predict the output of an imaging system if we know the **Input** and the **Characteristics** of the system. That is, \( g \) is the **Image** of the **Scene** \( f \).
In this chapter, functions are expressed with two dependent variables to represent a 2D image.

This convention is chosen to ensure consistency through the chapter, however the imaging problem can be treated in any number of dimensions.
4.2 IMAGE THEORY FUNDAMENTALS

4.2.1 Linear Systems Theory

The image, \( g(x,y) \), portrays a cross-section of the thorax, \( f(x,y) \), blurred by the transfer function, \( H \), of the imaging system:

\[
g(x, y) = H\{f(x,y)\}
\]
4.2 IMAGE THEORY FUNDAMENTALS

4.2.1 Linear Systems Theory

Unfortunately, this general approach to image analysis is very difficult to use.

It is necessary to compute the transfer function at Each Location in the image for each unique object or scene.

This analysis is greatly simplified when two fundamental assumptions can be made:

- **Linearity** and
- **Shift-Invariance**

abbreviated jointly as **LSI**
4.2 IMAGE THEORY FUNDAMENTALS
4.2.1 Linear Systems Theory

Linearity

A linear system is one in which the output of the system can be expressed as a **Weighted Sum** of the input constituents.

Thus, if a system presented with input $f_1$ results in output:

$$ g_1(x, y) = H\{f_1(x, y)\} $$

and input $f_2$ results in output:

$$ g_2(x, y) = H\{f_2(x, y)\} $$

then:

$$ H\{af_1(x, y) + bf_2(x, y)\} = H\{af_1(x, y)\} + H\{bf_2(x, y)\} $$

$$ = ag_1(x, y) + bg_2(x, y) $$
In general, most imaging systems are either

- Approximately linear or
- Can be linearized or
- Can be treated as being linear over a small range

The **Assumption of Linearity** lets us formulate the transfer function as an integral of the form

\[ g(x, y) = \iint f(x', y')H(x, y; x', y')\,dx'\,dy' \]
4.2 IMAGE THEORY FUNDAMENTALS

4.2.1 Linear Systems Theory

Linearity

However, most modern imaging systems are Digital.

As a result, images consist of measurement made at specific locations in a Regular Grid.

With Digital systems, these measurements are represented as an array of Discrete values.
In the Discrete case,

our expression can be reformulated as multiplication of a matrix $H$

where the input scene and output image are given as *Vectors* (for 1D images) or *Matrices* (for higher dimension images):

$$g = Hf$$
In this formulation, each element in $g$ is called a **Pixel** or **Picture Element**

Each element in $f$ is called a **Del** or **Detector Element**

A pixel represents the smallest region which can uniquely encode a single value in the image

By similar reasoning, the term **Voxel** or **Volume Element** is used in 3D imaging
4.2.1 Linear Systems Theory

Linearity

In the expression for the imaging system:

\[ g \] is expressed as a weighted sum, \( H \), of the source signals, \( f \)

It is important to note that \( H \) or \( H \) is still quite complicated

If \( g \) and \( f \) have \( m \times n \) elements

then \( H \) has \( (mn)^2 \) elements

that is, there is a Unique transfer function for each pixel in the image because the value of each pixel arises from a different weighted sum of the dels
A system is shift invariant if the system response function, \( H \), does not change as a function of position in the image.

By further adding the stipulation of shift-invariance, it is possible to formulate the transfer function without reference to a specific point of origin.

This allows us to write the integration in our expression as a convolution:

\[
g(x, y) = \iint f(x', y')h(x - x', y - y')dx'dy'
\]

where \( h \) is now a function of 2 variables while \( H \) was a function of 4 variables in the case of a 2D imaging system.
In the discrete formulation of a shift invariant system, the matrix $H$ now has a unique property; it is Toeplitz.

As a practical measure, we often use a circulant approximation of the Toeplitz matrix.

This approximation is valid provided the PSF is small compared to the size of the detector.
The discrete Fourier transform of the circulant approximation of $H$ is a diagonal matrix.

This property has particular appeal in analysing LSI systems, as we have gone from a formulation in which $H$ has:

as many as $(mn)^2$ non-zero elements to one that

has exactly $mn$ distinct elements.
4.2 IMAGE THEORY FUNDAMENTALS

4.2.1 Linear Systems Theory

Shift Invariance

As a result, it is possible to construct a new matrix, $h$, from $H$ such that our expression can now be rewritten

$$g = h \ast f$$

where $\ast$ is the circulant convolution operator.

In the case of 2D detectors and images $f$, $g$, and $h$ are each matrices with $m \times n$ distinct elements which are cyclically extended in each direction.
The assumptions of **Linearity** and **Shift-Invariance** are key to making most imaging problems tractable as there is now a **common** Transfer Function, $h$, that applies to each pixel in the image.
4.2 IMAGE THEORY FUNDAMENTALS

4.2.1 Linear Systems Theory

**Shift Invariance**

Recalling that for Fourier transform pairs the **Convolution** in one domain corresponds to **Multiplication** in the other domain, we can now rewrite the last expression as:

\[
\tilde{g} = \tilde{h} \tilde{f}
\]

where the tilde (\(\sim\)) denotes the discrete Fourier transform.

This implies that an object with a given spatial frequency referenced at the plane of the detector will result in an image with exactly the same spatial frequency, although the **Phase** and **Amplitude** may change.
Shift Invariance

With few exceptions most systems are not truly shift-invariant.

For Example

Consider a simple system in which a pixel in the image is equal to the Average of the matching del in the scene and the eight immediate neighbouring dels.

The transfer function will be Identical for all interior pixels.

However, pixels on the 4 Edges and 4 Corners of the image will have different transfer functions, because they do not have a Full Complement of neighbouring pixels upon which to calculate this average.
Shift Invariance

That said, most systems can be treated as shift invariant (with regard to this Boundary Problem), provided the blurring (or correlation) between pixels is small compared to the size of the image.

A Second strategy to ensure shift invariance is to consider the transfer function locally, rather than globally.

This strategy allows one to ignore differences in the detector physics across the full-field of the detector, such as the oblique incidence of X-rays.
4.2 IMAGE THEORY FUNDAMENTALS
4.2.2 Stochastic Properties

In all real imaging systems, it is necessary to consider the degradation of the image from both:

- **Blurring**, given by the transfer characteristics, and the
- **Presence of Noise**

Noise can arise from a number of sources, including the:

- **Generation** of the signal carriers,
- **Propagation** and **Transformation** of these carriers through the imaging process, and
- **Addition of Extraneous Noise** from various sources such as the imaging electronics
Thus, it is necessary to modify the image transfer equation to include a term for the noise, $n$

Noise is generated from a Random process.

As a result, the noise recorded in each image will be Unique.

Any given image $\hat{g}$ will include a single realization of the noise, $\hat{n}$, so that

$$\hat{g} = H\hat{f} + \hat{n}$$
Strictly speaking, some noise (e.g. X ray quantum noise) will be generated in the process of forming the scene, \( f \), and hence will be acted upon by the transfer function, \( H \)

while other noise (e.g. electronic readout noise) will not have been acted upon by the transfer function

**Equation Ignores This Distinction**
Also, strictly speaking, all quanta do not necessarily experience the same transfer function.

Variability in the transfer of individual quanta leads to the well-known Swank and Lubberts’ effects.
The introduction of noise in images means that imaging systems have to be evaluated *Statistically*.

The exact treatment of the images is dependent upon *Both* the nature of the noise present when the image is recorded and the imaging system.

System linearity (or *Linearizability*) will help to make the treatment of images in the presence of noise tractable.

In general, however, we also need to assume that the noise is *Stationary*.
4.2 IMAGE THEORY FUNDAMENTALS

4.2.2 Stochastic Properties

A stochastic noise process is **Stationary** if the process does not change when shifted either in time or in space.

That is, the **Moments** of a stationary process will not change based upon the time when observations begin.

An **Example** is X-ray quantum noise, because the probability of generating an X-ray does not depend upon when the previous or subsequent X-ray quanta are created.

Similarly, in a shift-invariant imaging system, it does not matter which point on the detector is used to calculate the moments of a stationary process, as each point is nominally the same.
4.2 IMAGE THEORY FUNDAMENTALS

4.2.2 Stochastic Properties

A Wide-Sense Stationary (WSS) process is one in which only the mean and covariance are stationary.

Since a Poisson process is fully characterized by the Mean and a Gaussian process is fully characterized by the Mean and Variance, it is typical to only require an imaging process to be WSS.

It is, in fact, common to treat the noise as being Gaussian and having Zero mean.

In practice, this is sufficient for Almost All imaging systems.
It should be noted that digital images consisting of *Discrete* arrays of pixels or volume elements (voxels) are not strictly stationary.

Shifts of the origin that are not commensurate with the pixel spacing will potentially result in different images being acquired.

However, a system is said to be *Cyclostationary* if the statistical properties are unchanged by shifts in the origin of specific amounts, i.e. multiples of the pixel or voxel pitch.
A system is **Wide-Sense Cyclo-stationary** if the mean and covariance are unchanged by specific shifts in the origin.

In general, we can assume most digital imaging systems are wide-sense cyclo-stationary, at least **Locally**.
4.2 IMAGE THEORY FUNDAMENTALS

4.2.2 Stochastic Properties

To measure the signal in a pixel, exclusive of the noise, we may simply **Average** the value in that pixel over many images to minimize the influence of the noise on the measurement.

In a similar fashion, we can estimate the noise in a pixel by calculating the **Standard Deviation** of the value of that pixel over many images of the same scene.

Calculations which involve a large number of images are clearly **Time-Consuming** to acquire and process in order to estimate the mean and standard deviation with **Sufficient Accuracy**.
However this problem is tremendously simplified if one can additionally assume **Ergodicity**

**Ergodic Process**: one in which the statistical properties of the ensemble can be obtained by analysing a single realization of the process.

For example, X ray quantum noise is frequently referred to as **White Noise**, implying that:

- In different realizations all spatial frequencies are equally represented, or *equivalently* that
- The noise from individual quanta are uncorrelated.
White Noise is Ergodic
This means, for example, that we can calculate the average fluence of an X ray beam either by averaging over a Region or averaging over Multiple Images

When an appropriate imaging system is used to image an ergodic process (such as a uniform scene imaged with X rays), calculations performed from a number of sample images can be replaced by calculations from One Image.

For Example, the noise in a particular pixel that was originally measured from image samples can now be measured from a region of a single image.
With few exceptions (notably screen-film radiography), modern imaging systems are Digital.

A digital image is only defined as discrete points in space, called sampling points.

The process of sampling by a detector element (del) generally involves the integration of continuous signal values over a finite region of space around the sampling point.

The Shape of these regions is defined by the Sampling Aperture.

Distance between sampling points is called the Sampling Pitch.
In an idealized 2D detector, the sampling aperture of each del is represented by a square of dimension, $a'$.

Such dels are repeated with pitch $a$ to cover the entire detector:

Rectangular array of dels in which a single del with a square aperture of dimensions $a' \times a'$ is shown centred upon a series of sampling points with pitch $a$ in orthogonal directions.
It is not strictly necessary for the aperture and pitch to have the same size, nor to be square.

For example, active matrix X-ray detectors can have regions which are not radiation sensitive such as the data and control lines and del readout electronics.

The Fill Factor of an active matrix detector is typically defined as the ratio $a'^2/a^2$.

The Fill Factor is commonly $<1$. 
It is also possible for the del aperture to be $>a^2$

For Example, in CR, the scanning laser will typically stimulate fluorescence from a circular region having a diameter greater than the sampling pitch.

As discussed later, this has benefit in Reducing Aliasing.
The process of sampling a continuous signal $f$ by a single del is given by:

$$f(x_i, y_j) = \iint f(x, y)A(x - x_i a, y - y_j a) \, dx \, dy$$

where $A$ is the aperture function and $(x_i, y_j)$ are integer indices of the del.

In practice, the aperture function is Non-Zero over a limited area thus providing finite limits to the integral in this equation.
It is clear from this expression that if one were to shift the sampling points by a non-integer amount (i.e. incommensurate with the pixel pitch), the recorded image would vary.

It is for this reason that digital systems are only Cyclo-Stationary.

In general these changes are small – especially for objects which are Large relative to the sampling pitch.

However, for small objects, these changes can be significant.
Sampling a continuous signal $f(x, y)$ on a regular grid with grid spacing $a$, is equivalent to multiplying $f$ by a comb function, $\text{comb}_a$.

The comb function is an infinite sum of Dirac delta functions centred at the sampling points.

Multiplication by the comb function in the image domain is equivalent to Convolution by the Fourier transform (FT) of the comb function in the Fourier domain.
The FT of the comb function is also a comb function, but with grid spacing $1/a$.

This convolution has the form:

$$(\mathcal{F} * \text{comb}_{1/a})(u, v) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \mathcal{F} \left( u - \frac{j}{a}, v - \frac{k}{a} \right)$$

This implies that the FT of $f$ is replicated at each point on a grid with a spacing $1/a$, and an infinite sum of all the replicates is taken.
The frequency $1/a$ is called the **Sampling Rate**

The *Nyquist-Shannon* sampling theorem provides **Guidance** in determining the value of $a$ needed for a specific imaging task:

Ideally, the Fourier spectrum of $f$ should not have components above the frequency $1/2a$

This frequency is called the **Nyquist Frequency**
When this condition is not met, the Fourier spectra will contain components with spatial frequencies which **Exceed** the Nyquist frequency, and the infinite sum of spectra will overlap.
This overlap between the superimposed spectra will result in **Aliasing**.

Aliasing degrades the sampled image because it **incorrectly** portrays high-frequency information present in the scene as lower-frequency information in the image. The **Black Curve** in figure demonstrates this effect.

To avoid aliasing, the Nyquist frequency must be greater than or equal to the maximum frequency in the image prior to sampling.

In many system designs, it is **impossible** to avoid aliasing.
Contrast is defined as the ratio of the signal difference to the average signal.

The rationale behind this is that a small difference is negligible if the average signal is **Large**, while the same small difference is readily visible if the average signal is **Small**.

In general, in medical imaging, we will want to achieve the **highest** contrast possible to best visualize disease features.
There are two common definitions of contrast in medical imaging.

The **Weber Contrast**, or the Local Contrast, is defined as:

\[
C = \frac{f_f - f_b}{f_b}
\]

where \(f_f\) and \(f_b\) represent the signal of the feature and the background, respectively.
4.3 CONTRAST

4.3.1 Definition

Note: Contrast is defined in terms of the scene $f$

As we will see, it is equally acceptable to consider the contrast:

- Of the image $g$, or
- Measured at other points in the image chain
  such as the contrast of a feature displayed on a computer monitor

The **Weber Contrast** is commonly used in cases where small features are present on a large uniform background
4.3 CONTRAST

4.3.1 Definition

The Modulation or Michelson Contrast is commonly used for patterns where both bright and dark features take up similar fractions of the image.

The Modulation Contrast is defined as:

\[ C_M = \frac{f_{\text{max}} - f_{\text{min}}}{f_{\text{max}} + f_{\text{min}}} \]

where \( f_{\text{max}} \) and \( f_{\text{min}} \) represent the highest and lowest signals.
4.3 CONTRAST
4.3.1 Definition

The Modulation Contrast has particular interest in the Fourier analysis of medical images

Consider a signal of the form:

\[ f(x, y) = A + B \sin(2\pi ux) \]

Substituting into the Modulation Contrast gives:

\[ C_M = \frac{A + B - (A - B)}{A + B + A - B} = \frac{B}{A} \]

Thus, we see that the **Numerator** expresses the amplitude or difference in the signal \( B = (f_{\text{max}} - f_{\text{min}})/2 \), while the **Denominator** expresses the average signal \( A = (f_{\text{max}} + f_{\text{min}})/2 \)
Care should be taken as to which definition of contrast is used.

The correct choice is situation dependent.

In general, the **Local Contrast** is used when a small object is presented on a uniform background, such as in simple observer experiments (e.g., 2-AFC experiments).

The **Modulation Contrast** has relevance in the Fourier analysis of imaging systems.
4.3 CONTRAST

4.3.2 Contrast Types

In medical imaging, the **Subject Contrast** is defined as the contrast (whether local or modulation) of the object in the scene being imaged.

**For example:**

- **In X ray Imaging**, the subject contrast depends upon the X ray spectrum, and the attenuation of the object and background.

- **In Radionuclide Imaging**, the subject contrast depends upon radiopharmaceutical uptake by the lesion and background, the pharmacokinetics, and the attenuation of the gamma rays by the patient.

Similarly, one can define the subject contrast for CT, MRI and ultrasound.
4.3 CONTRAST

4.3.2 Contrast Types

The Image Contrast depends upon the subject contrast and the characteristics of the imaging detector.

For example:

In Radiographic Imaging, the image contrast is affected by:

- the X Ray Spectrum incident upon the X ray converter (e.g. the phosphor or semiconductor material of the X ray detector);
- the converter Composition and Thickness, and
- the Grayscale Characteristics of the convertor, whether analogue or digital.
The **Display Contrast** is the contrast of the image as displayed for final viewing by an observer.

The Display Contrast is dependent upon:

- the **Image Contrast** and
- the **Grayscale Characteristics** of the display device and
- any **Image Processing** that occurs prior to or during display.
In the absence of blurring, the ratio of the image contrast to the display contrast is defined as the **Transfer Function** of the imaging system.

The grayscale response of **Film** is non-linear.

Thus, to stay within the framework of LSI systems analysis, it is necessary to **Linearize** the response of the film.
This is typically done using a small-signals model in which the low-contrast variations in the scene recorded in the X-ray beam, $\Delta I/I_0$, produce linear changes in the film density, $\Delta D$, such that

$$\Delta D = \gamma \frac{\log_{10}(e) \Delta I}{I_0}$$

where $\gamma$ is called the **Film Gamma** and typically has a value of between 2.5 and 4.5.
Two grayscale response functions are shown:
The Grayscale Characteristic, $\Gamma$, can now be calculated as:

$$\Gamma = \frac{\Delta D}{\Delta I} = \frac{\gamma \log_{10} e}{I_0}$$
In a similar fashion, the grayscale characteristic of a digital system with a Digital Display can be defined.

In general, digital displays have a non-linear response with a Gamma of between 1.7 and 2.3.
4.3 CONTRAST

4.3.3 Grayscale Characteristics

It should be noted that \( \Gamma \) does not consider the spatial distribution of the signals.

In this sense, we can treat \( \Gamma \) as the response of a detector which records the incident X ray quanta, but does not record the Location of the X ray quanta.

Equivalently, we can consider it as the DC (static) response of the imaging system.
Given that the Fourier transform of a constant is equal to a Delta Function at zero spatial frequency we can also consider this response to be the

**Zero Spatial Frequency Response**

of the imaging system
4.4 UNSHARPNESS

In the preceding discussion of contrast, we considered Large Objects in the Absence of blurring.

However, in general, we cannot ignore either assumption.

When viewed from the spatial domain, blurring reduces contrast of small objects.

The effect of blurring is to spread the signal laterally, so that a focused point is now a Diffuse point.
One fundamental property of blurring is that the more the signal is spread out, the lower its intensity, and thus the lower the contrast.

An image of a point is shown blurred by convolution with a Gaussian kernel of diameter 16, 32 and 64 pixels.
This also means that the peak signal is only degraded if the size of the object is **smaller** than the width of the blurring function.

The contrast of larger objects will not be affected:
Consider the operation of an **Impulse Function** on an imaging system

If an imaging system is characterized by a LSI response function \( h(x - x', y - y') \), then this response can be measured by providing a **Delta Function** as input to the system.

Setting \( f(x, y) = \delta(x, y) \) gives:

\[
g(x, y) = \iint \delta(x, y) h(x - x', y - y') dx' dy' = h(x, y)
\]
4.4 UNSHARPNESS

4.4.1 Quantifying Unsharpness

We refer to the system transfer function as the point spread function, **PSF**, when specified in the spatial domain.

In fact, the blurring of a point object, seen in images, is a pictorial display of the PSF.

It is common to consider the PSF either as being:

- **Separable**
  
  
  \[ h(x, y) = h(x)h(y) \]

- **Circular Symmetric**

  \[ h(r) = h(x, y) \]

where

\[ r = \sqrt{x^2 + y^2} \]

depending upon the imaging system.
While it is possible to calculate the blurring of any object in the spatial domain via convolution with the LSI system transfer function, $h$, the problem is generally better approached in the Fourier Domain. To this end, it is informative to consider the effect of blurring on Modulation Contrast. Consider a sinusoidal modulation given by:

$$f(x, y) = A + B\sin(2\pi(ux + vy))$$
The recorded signal will be **degraded** by the system transfer function

\[ \tilde{h}(u, v) \]

such that

\[ g(x, y) = A\tilde{h}(0,0) + B|\tilde{h}(u, v)|\sin(2\pi(ux + vy)) \]

Here, any **Phase Shift** of the image relative to the scene is ignored for simplicity.

We see, therefore, that the modulation contrast of object \( f \) is

\[ C_f = \frac{B}{A} \]
4.4 UNSHARPNESS

4.4.1 Quantifying Unsharpness

The Modulation Contrast of the image, \( g \), is:

\[
C_g = \frac{B|\tilde{h}(u,v)|}{A\tilde{h}(0,0)}
\]

We can now define a new function, \( T \), called the Modulation Transfer Function (MTF), which is defined as the absolute value ratio of \( C_g/C_f \) at a given spatial frequency \( (u, v) \):

\[
T(u, v) = \frac{|\tilde{h}(u,v)|}{\tilde{h}(0,0)}
\]
4.4 UNSHARPNESS

4.4.1 Quantifying Unsharpness

The MTF quantifies the degradation of the contrast of a system as a function of spatial frequency.

By definition, the modulation at zero spatial frequency, \( T(0,0) = 1 \).

In the majority of imaging systems, and in the absence of image processing, the MTF is bounded by \( 0 \leq T \leq 1 \).

In addition, it should also be noted that based on the same derivation, the Grayscale Characteristic:

\[
\Gamma = \tilde{h}(0,0)
\]
4.4 UNSHARPNESS

4.4.1 Quantifying Unsharpness

The measurement of the 2D PSF for projection or cross-sectional imaging systems or 3D PSF for volumetric imaging systems (and hence the corresponding 2D or 3D MTF) requires that the imaging system be presented with an **Impulse Function**.

In practice, this can be accomplished by imaging a pinhole in radiography, a wire in cross-section in axial CT, or a single scatterer in ultrasound.

The knowledge of the MTF in 2D or 3D is useful in calculations in **Signal Detection Theory**.
4.4 UNSHARPNESS

4.4.1 Quantifying Unsharpness

It is more common, however, to measure the MTF in a Single dimension.

In the case of Radiography, a practical method to measure the 1D MTF is to image a slit formed by two metal bars spaced closely together.

Such a slit can be used to measure the LSF.

Among other benefits, imaging a slit will provide better resilience to quantum noise, and multiple slit camera images can be superimposed (Boot-Strapped) to better define the tails of the LSF.
The LSF is, in fact, an **Integral Representation** of the 2D PSF.

For example, consider a slit aligned vertically in an image which here we assume corresponds to the y-axis.

Then the LSF, \( h(x) \) is given by:

\[
h(x) = \int h(x,y) \, dy
\]

The integral can be simplified if we assume that the PSF is **Separable**:

\[
\int h(x)h(y) \, dy = h(x)
\]

as in video-based imaging systems.
4.4 UNSHARPNESS

4.4.1 Quantifying Unsharpness

It should be clear from this that the LSF and the 1D MTF are Fourier transform *pairs*

If we assume a **Rotationally Symmetric** PSF, as might be found in a phosphor-based detector, the PSF is related to the LSF by the **Abel transform**:

\[
h(x) = 2 \int_x^\infty \frac{h(r)r}{\sqrt{x^2-r^2}} \, dr
\]

and

\[
h(r) = -\frac{1}{\pi} \frac{d}{dr} \left( \int_r^\infty \frac{h(x)\, dx}{x\sqrt{x^2-r^2}} \right)
\]
Note that while the forward Abel transform is *Tractable*, the inverse transform is not.

However, the inverse transform can be calculated by first applying the Fourier transform and then the *Hankel* transform:

The 1D forms of the system response function are shown, together with the functional relationship.
4.4 UNSHARPNESS

4.4.1 Quantifying Unsharpness

A further **Simplification** is to image an **Edge**, rather than a line

The **Edge Spread Function** (ESF) is simply an integral representation of the LSF, so that:

\[ e(x) = \int_{-\infty}^{x} h(x) \, dx \]

and

\[ h(x) = \frac{d}{dx} e(x) \]
Today, the ESF is the **Preferred Method** for measuring the system response function of radiographic systems.

There are **Two** clear benefits:

- an edge is **Easy** to produce for almost any imaging system, although issues such as the position of the edge need to be carefully considered
- the ESF is amenable to measuring the **Pre-Sampled MTF** of digital systems
4.4 UNSHARPNESS

4.4.2 Measuring Unsharpness

Limiting Spatial Resolution

The spatial resolution is a metric to quantify the ability of an imaging system to display two unique objects closely separated in space.

The limiting spatial resolution is typically defined as the maximum spatial frequency for which modulation is preserved without distortion or aliasing.
4.4 UNSHARPNESS
4.4.2 Measuring Unsharpness

Limiting Spatial Resolution

The limiting resolution can be measured by:

- Imaging line patterns or star patterns in radiography and
- Arrays of cylinders imaged in cross-section in cross-sectional imaging systems such as CT and ultrasound

All of these methods use high-contrast, sharp-edged objects
4.4 UNSHARPNESS

4.4.2 Measuring Unsharpness

Limiting Spatial Resolution

As such the limiting spatial resolution is typically measured in **Line Pairs** per unit length.

This suggests that the basis functions in such patterns are **Rect Functions**.

By contrast, the MTF is specified in terms of **Sinusoids**.

This is specified in terms of spatial frequencies in **Cycles** per unit length.
There is no strict relationship between a particular MTF value and the limiting spatial resolution of an imaging system.

The **Coltman Transform** can be used to relate the:

- **Square Wave Response** measured with a bar or star pattern
- and
- the Sinusoidal Response measured by the **MTF**
Ultimately, however, the ability to detect an object (and hence resolve it from its neighbour) is related to the **Signal to Noise Ratio** of the object.

As a **Rule of Thumb**, the limit of resolution for most imaging systems for high-contrast objects (e.g., a bar pattern) occurs at the spatial frequency where the

\[ MTF \approx 0.05 \ (5\%) \]
4.4 UNSHARPNESS

4.4.2 Measuring Unsharpness

Modulation Transfer Function (MTF)

In practice, it is difficult to measure the MTF of an analogue system (such as film) without first digitizing the analogue image.

As such, it is important that the digitization process satisfies the Nyquist-Shannon sampling theorem to avoid aliasing.

This is possible in some instances, such as digitizing a film, where the digitizer optics can be designed to eliminate aliasing.
4.4 UNSHARPNESS
4.4.2 Measuring Unsharpness

Modulation Transfer Function (MTF)

In this instance, however, the MTF that is measured is not the MTF of the film but rather is given by:

\[ T_m = T_a T_d \]

where \( T_m \) is the measured MTF, \( T_a \) is the MTF of the analogue system, and \( T_d \) is the MTF of the digitizer.

With this equation, it is possible to recover \( T_a \) provided \( T_d > 0 \) over the range of frequencies of interest.
4.4 UNSHARPNESS

4.4.2 Measuring Unsharpness

Modulation Transfer Function (MTF)

In many systems, however, it is **not** possible to avoid aliasing.

For example, in a DR detector that consists of an a-Se photoconductor coupled to a TFT array.

The Selenium has Very High Limiting Spatial Resolution

much higher than can be supported by the pixel pitch of the detector.
Modulation Transfer Function (MTF)

This resolution pattern is made with such a system:

A digital radiograph of a bar pattern is shown

Each group in the pattern (e.g. 0.6 lp/mm) contains three equally spaced elements
4.4 UNSHARPNESS

4.4.2 Measuring Unsharpness

Modulation Transfer Function (MTF)

A magnified region of the pattern is shown:

Here, we can deduce that the limiting resolution is 3.4 lp/mm

Higher frequencies are aliased as shown by the reversal of the bands (highlighted in yellow) which arise from the digital sampling process.
4.4 UNSHARPNESS

4.4.2 Measuring Unsharpness

Modulation Transfer Function (MTF)

In such instances, there are some important facts to understand. **First**, aliasing will occur with such a system, as can be seen. **It is Unavoidable**

This means, that predicting the exact image recorded by a system requires knowledge of the:

- **Location** of the objects in the scene relative to the detector matrix with sub-pixel precision, as well as the
- **Blurring** of the system prior to sampling
4.4 UNSHARPNESS
4.4.2 Measuring Unsharpness

Modulation Transfer Function (MTF)

The latter can be determined by measuring what is known as the Pre-Sampling MTF.

The pre-sampling MTF is measured using a high sampling frequency so that No Aliasing is present in the measurement.

It is important to realise that in spite of its name, the pre-sampling MTF does include the blurring effects of the sampling aperture.
4.4 UNSHARPNESS
4.4.2 Measuring Unsharpness

Modulation Transfer Function (MTF)

The pre-sampling MTF measurement starts with imaging a well-defined edge placed at a small angle (1.5°- 3°) to the pixel matrix/array. From this digital image, the exact angle of the edge is detected and the distance of individual pixels to the edge is computed to construct a Super-Sampled (SS) edge spread function (ESF). Differentiation of the SS-ESF generates a LSF, whose FT gives the MTF.

This is the Preferred Method for Measuring the MTF Today
In the previous section, we dealt with the special situation in which an analogue image, such as film, is digitized by a device such as a scanning photometer.

In this situation, the measured MTF is the **Product** of the film MTF and the MTF of the scanning system.

This principle can be extended to more generic imaging systems which are composed of a **Series** of individual components.
Example of how a system MTF is the product of its components:

The overall or system MTF is the product of the MTFs of the three components A, B and C.
4.4 UNSHARPNESS

4.4.3 Resolution of a Cascaded Imaging System

A **Classic Example** is to compare the blurring of the focal spot and imaging geometry with that of the detector.

Another classic example is of a video fluoroscopic detector containing an X ray image intensifier.

In this instance, the MTF of the image is determined by the MTFs of the:

- Image intensifier
- Video camera
- Optical coupling
This is true because the image passes sequentially through each of the components, and each successive component sees an increasingly blurred image.

The **One Caveat** to this concept is that aliasing must be addressed very carefully once sampling has occurred.

The principle of **Cascaded Systems Analysis** is frequently used, as it:

- Allows one to determine the impact of each component on spatial resolution, and
- Provides a useful tool for analysing how a system design can be improved.
4.5 NOISE

The Greek philosopher Heraclitus (c. 535 B.C.) is claimed to have said that:

“You cannot step twice into the same river"

It can similarly be asserted that one can never acquire the same image twice

There Lies the Fundamental Nature of Image Noise
Noise arises as **Random** variations in the recorded signal (e.g. the number of X-ray quanta detected) from pixel-to-pixel.

**Noise is Not Related to Anatomy**

Rather, it arises from the **random** generation of the image signal.

**Note**, however, that noise is related for example to the number of X-ray quanta; thus, highly attenuating structures (like bones) will look noisier than less attenuating structures.
4.5 NOISE

In a well-designed X-ray imaging system, X-ray quantum noise will be the **Limiting Factor** in the detection of objects.

The ability to detect an object is dependent upon both the contrast of the object and the noise in the image.

As illustrated, the ability to discern the disk is degraded as the magnitude of the noise is increased.
4.5 NOISE

The optimal radiation dose is just sufficient to visualize the anatomy or disease of interest, thus minimizing the potential for harm.

In a seminal work, **Albert Rose** showed that the ability to detect an object is related to the ratio of the signal to noise.

We shall return to this important result.

However, first we must learn the **Fundamentals** of image noise.
4.5 NOISE

4.5.1 Poisson Nature of Photons

The process of generating X-ray quanta is **Random**

The intrinsic fluctuation in the number of X-ray quanta is called **X-ray Quantum Noise**

X-ray quantum noise is **Poisson** distributed

In particular, the probability of observing $n$ photons given $\alpha$, the mean number of photons, is

$$P(n, \alpha) = \frac{\alpha^n e^{-\alpha}}{n!}$$

where $\alpha$ can be any positive number and $n$ must be an integer
4.5 NOISE

4.5.1 Poisson Nature of Photons

A fundamental principle of the Poisson distribution is that the variance, $\sigma^2$, is **Equal** to the mean value, $\alpha$

When dealing with large mean numbers, most distributions become approximately **Gaussian**

This applies to the Poisson distribution when a large number of X-ray quanta (e.g. >50 per del) are detected
The **Mean-Variance Equality** for X-ray quantum noise limited systems is useful experimentally.

**For Example**, it is useful to test whether the images recorded by a system are limited by the X-ray quantum noise.

Such systems are said to be X-ray quantum noise limited, and the X-ray absorber is called the **Primary Quantum Sink** to imply that the **Primary** determinant of the image noise is the **Number** of X-ray quanta recorded.
4.5 NOISE

4.5.1 Poisson Nature of Photons

In the mean-variance experiment, one measures the mean and standard deviation **Parametrically** as a function of dose.

When plotted **Log-Log**, the slope of this curve should be \(1/2\).

When performed for digital X-ray detectors, including CT systems, this helps to determine the range of air kerma or detector dose over which the system is X-ray **Quantum Noise Limited**.
Image noise is said to be **Uncorrelated** if the value in each pixel is independent of the values in neighbouring pixels.

If this is true and the system is **Stationary** and **Ergodic**, then it is trivial to achieve a complete characterization of the system noise.

One simply needs to calculate the **Variance** (or Standard Deviation) of the image on a per-pixel basis.
Uncorrelated noise is called **White Noise** because all spatial frequencies are represented in equal amounts.

All X-ray noise in images starts as white noise, since the production of X-ray quanta is **Uncorrelated** both in time and in space. Thus, the probability of creating an X-ray at any point in time and any particular direction does not depend on the previous quanta which were generated, nor any subsequent quanta.
Unfortunately, it is **Rare** to find an imaging system in which the resultant images are uncorrelated in space.

This arises from the fact that each X-ray will create multiple **Secondary Carriers** which are necessarily correlated, and these carriers diffuse from a single point of creation.

Thus the signal recorded from a single X-ray is often **Spread** among several pixels.

As a result the pixel variance is reduced and neighbouring pixel values are **Correlated**.
Noise can also be correlated by spatial non-uniformity in the imaging system - that is, **Non-Stationarity**

In most real imaging systems, the condition of stationarity is only **Partially** met.

One is often placed in a situation where it must be decided if the stationarity condition is sufficiently met to treat the system as **Shift Invariant**.
An Example:

An early digital X-ray detector prototype is shown which consisted of a phosphor screen coupled to an array of fibre-optic tapers and CCD cameras.

The image on the right is a measurement of the per-pixel variance on a small region indicated by yellow on the detector face.
4.5 NOISE

4.5.2 Measures of Variance & Correlation/Co-variance

The image on the right is a **Variance Image**, obtained by estimating the variance in each pixel using multiple images, i.e. multiple realizations from the ensemble.

The image shows that there are strong spatial variations in the variance due to:

- Differences in the **Coupling Efficiency** of the fibre optics and the
- **Sensitivity** differences of the CCDs
Noise can be characterized by the **Auto Correlation** at each point in the image, calculated as the ensemble average:

$$R(x, y, x + \Delta x, y + \Delta y) = \langle \hat{g}(x, y)\hat{g}(x + \Delta x, y + \Delta y) \rangle$$

Here, we use the notation $\hat{g}$ to denote that $g$ is a random variable.

**Correlations** about the mean

$$\Delta\hat{g}(x, y) = \hat{g}(x, y) - \langle \hat{g}(x, y) \rangle$$

are given by the **Autocovariance Function**

$$K(x, y, x + \Delta x, y + \Delta y) = \langle \Delta\hat{g}(x, y)\Delta\hat{g}(x + \Delta x, y + \Delta y) \rangle$$
Based on the assumption of **Stationarity**, 
\[
\langle \hat{g}(x, y) \rangle = g,
\]
is a constant independent of position.

If the random process is **Wide-Sense Stationary**, then both the autocorrelation and the autocovariance are independent of position \((x, y)\) and only dependent upon displacement.

\[
R(\Delta x, \Delta y) = R(x, y, x + \Delta x, y + \Delta y)
\]
\[
K(\Delta x, \Delta y) = K(x, y, x + \Delta x, y + \Delta y)
\]
If the random process is **Ergodic**, then the ensemble average can be replaced by a spatial average.

Considering a digital image of a stationary ergodic process, such as incident X-ray quanta, the autocovariance forms a matrix.

\[
K(\Delta x, \Delta y) = \frac{1}{2X} \frac{1}{2Y} \sum_{-X}^{+X} \sum_{-Y}^{+Y} g(x, y)g(x + \Delta x, y + \Delta y)
\]

where the region over which the calculation is applied is \(2X \times 2Y\) pixels.
The value of the **Autocovariance** at the origin is equal to the variance:

\[
K(0,0) = \langle \Delta g(x, y) \Delta g(x, y) \rangle = \sigma_A^2
\]

where the subscript \( A \) denotes that the calculation is performed over an aperture of area \( A \), typically the pixel aperture.
The correlation of noise can be determined in either the:

- **Spatial Domain** using autocorrelation (as we have seen in the previous section) or
- **Spatial Frequency Domain** using Noise Power Spectra (**NPS**) also known as **Wiener Spectra**
There are a number of requirements which must be met for the NPS of an imaging system to be tractable. These include: **Linearity**, **Shift-Invariance**, **Ergodicity** and **Wide-Sense Stationarity**.

In the case of digital devices, the latter requirement is replaced by **Wide-Sense Cyclo-Stationarity**.

If the above criteria are met, then the NPS completely describes the noise properties of an imaging system.
In point of fact, it is **Impossible** to meet all of these criteria **Exactly**

**For Example**, all practical detectors have finite size and thus are not strictly stationary

However, in spite of these limitations, it is generally possible to calculate the **Local NPS**
By **Definition**, the NPS is the ensemble average of the square of the Fourier transform of the spatial density fluctuations.

\[
W(u, v) = \langle \lim_{X,Y \to \infty} \frac{1}{2X} \frac{1}{2Y} \left| \int_{-X}^{+X} \int_{-Y}^{+Y} \Delta g(x, y) e^{-2\pi i (ux + vy)} \, dx \, dy \right|^2 \rangle
\]

The NPS and the autocovariance function form a **Fourier Transform Pair**.

This can be seen by taking the Fourier transform of the autocovariance function and applying the convolution theorem.
The NPS of a discrete random process, such as when measured with a Digital X-ray detector, is:

\[
W(u, v) = \langle \lim_{N_x N_y \to \infty} \frac{x_0 y_0}{N_x N_y} \left| \sum_{x,y} \Delta g(x, y) e^{-2\pi i (ux + vy)} \right|^2 \rangle
\]

This equation requires that we perform the summation over all space.

In practice, this is impossible as we are dealing with detectors of limited extent.

By restricting the calculation to a finite region, it is possible to determine the Fourier content of the fluctuations in that specific region.
We call this simple calculation a **Sample Spectrum**

It represents one possible instantiation of the noise seen by the imaging system, and we denote this by:

\[
\hat{W}(u, v) = \frac{x_0 y_0}{N_x N_y} \left| \sum_{m,n} \Delta g(x, y) e^{-2\pi i (umx_0 + vny_0)} \right|^2
\]

An estimate of the true NPS is created by **averaging** the sample spectra from \(M\) realizations of the noise

\[
\bar{W}(u, v) = \frac{1}{M} \sum_{i=1}^{M} \hat{W}_i(u, v)
\]
4.5 NOISE

4.5.3 Noise Power Spectra

**Ideally**, the average should be done by calculating sample spectra from **Multiple Images** over the same region of the detector.

However, by assuming Stationarity and Ergodicity, we can take averages over **Multiple Regions** of the detector, significantly reducing the number of images that we need to acquire.
Now, the estimate of the NPS, $\hat{W}$, has an accuracy that is determined by the number of samples used to make the estimate.

Assuming Gaussian statistics, at frequency $(u, v)$, the error in the estimate $\hat{W}(x, y)$ will have a standard error given by:

$$\sqrt{\frac{c}{M}} \hat{W}(u, v)$$

where $c=2$ for $u=0$ or $v=0$, and $c=1$ otherwise.

The values of $c$ arise from the circulant nature of the Fourier transform.
Typically, \(64 \times 64\) pixel regions are sufficiently large to calculate the NPS.

Approximately \(1000\) such regions are needed for good 2-D spectral estimates.

Remembering that the autocorrelation function and the NPS are Fourier transform pairs, it follows from Parseval’s Theorem that

\[
K(0,0) = \frac{1}{x_0 y_0 N_x N_y} \sum_{u,v} \hat{W}(u, v)
\]

This provides a useful and rapid method of verifying a NPS calculation.
There are Many Uses of the NPS

It is most commonly used in characterizing imaging device Performance

In particular, the NPS is exceptionally valuable in investigating Sources of detector noise

For Example, poor grounding often causes line-frequency (typically 50 or 60 Hz) noise or its harmonics to be present in the image

NPS facilitates the identification of this noise
4.5 NOISE

4.5.3 Noise Power Spectra

In such applications, it is common to calculate

**Normalized Noise Power Spectra (NNPS)**

since the absolute noise power is less important than the relative noise power

As we shall see, absolute calculations of the NPS are an integral part of **DQE** and **NEQ** measurements, and the NPS is required to calculate the **SNR** in application of signal-detection theory.
4.5 NOISE

4.5.3 Noise Power Spectra

Unlike the MTF, there is no way to measure the Pre-Sampling NPS.

As a result, high frequency quantum noise (frequencies higher than supported by the sampling grid) will be aliased to lower frequencies in the same way that high frequency signals are aliased to lower frequencies.

Radiation detectors with high spatial resolution, such as a-Se Photoconductors, will naturally alias high frequency noise.
Radiation detectors based on **Phosphors** naturally blur both the signal and the noise prior to sampling.

And thus can be designed so that both signal and noise aliasing are not present.

There is **No Consensus** as to whether noise aliasing is beneficial or detrimental.

Ultimately, the role of noise aliasing is determined by the imaging task, as we shall see later.
4.5 NOISE

4.5.3 Noise Power Spectra

As with the MTF, it is sometimes preferable to display 1D sections through the 2D (or 3D) noise power spectrum or autocovariance.

There are two presentations which are used:

the **Central Section**

\[ W_c(u) = W(u, 0) \]

and

the **Integral Form**

\[ W_I(u) = \sum_v W(u, v) \]
Similarly, if the noise is **Rotationally Symmetric**, the noise can be averaged in annular regions and presented radially.

The choice of presentation depends upon the intended use.

It is most **Common** to present the central section.

Regardless, the various 1D presentations are easily related by the central slice theorem, as shown in the next slide.
4.5 NOISE

4.5.3 Noise Power Spectra

Both 1D integral and central sections of the NPS and autocovariance can be presented.

The various presentations are related by integral (or discrete) transformations.

Here, the relationships for rotationally symmetric 1D noise power spectra and autocovariance functions are shown.
4.5 NOISE

4.5.4 NPS of a Cascaded Imaging System

The propagation or cascade of noise is substantially more complicated than the composition of the MTF.

A proper analysis of noise must account for the **Correlation** of the various noise sources.

These can be numerous, including:

- The primary X-ray **Quantum Noise**
- The noise arising from the **Transduction** of the primary quanta into secondary quanta (such as light photons in a phosphor or carriers in a semiconductor)
- Various **Additive Noise Sources** such as electronic noise from the readout circuitry of digital detectors
4.5 NOISE
4.5.4 NPS of a Cascaded Imaging System

While the general theory of noise propagation is beyond the scope of this work, the two simple examples which follow may be illustrative:

- **Image Subtraction**
- **Primary & Secondary Quantum Noise**
It is common to **Add** or **Subtract** or otherwise manipulate medical images.

A classic example is **Digital Subtraction Angiography (DSA)**.

In DSA, a projection image with contrast agent is subtracted from a pre-contrast mask image to produce an image that shows the **Difference** in attenuation between the two images which arises from the contrast agent.
4.5 NOISE

4.5.4 NPS of a Cascaded Imaging System

Image Subtraction

Strictly speaking, the Logarithms are subtracted

In the absence of patient motion, the resultant image depicts the

Contrast Enhanced Vascularity
Image Subtraction

The effect of the subtraction is to **increase** the image noise.

This arises because for a given pixel in the image, the pixel values in the mask and the contrast-enhanced images are **uncorrelated**.

As a result, the subtraction incorporates the noise of **both** images.
Image Subtraction

Noise adds in Quadrature, thus the noise in the subtracted image is $\sqrt{2}$ larger than the noise is in the source images.

To Ameliorate the noise increase in the subtraction image, it is typical to

Acquire the Mask Image at Much Higher Dose

thereby reducing the contribution of the mask noise to the subtraction.
Primary & Secondary Quantum Noise

Consider this simple imaging system:

The concept of Quantum Accounting is illustrated.

A simple X-ray detector is shown.

At each stage of the imaging system, the number of quanta per incident X-ray is calculated to determine the Dominant noise source.
4.5 NOISE

4.5.4 NPS of a Cascaded Imaging System

Primary & Secondary Quantum Noise

In this system:

- X-ray quanta are incident on a phosphor screen (Stage 1)
- A fraction of those quanta are absorbed to produce light (Stage 2)
- A substantial number of light quanta (perhaps 300-3000) are produced per X-ray quantum (Stage 3)
- A small fraction of the light quanta are collected by the lens (Stage 4)
- A fraction of the collected light quanta produce carriers in the optical image receptor (e.g. a CCD camera) (Stage 5)
The process of producing an electronic image from the source distribution of X-rays will necessarily introduce noise. In fact, each stage will alter the noise of the resultant image.

In this simple model, there are two primary sources of noise:

- X-ray (or Primary) quantum noise
- Secondary quantum noise
4.5 NOISE

4.5.4 NPS of a Cascaded Imaging System

Primary & Secondary Quantum Noise

Secondary Quantum Noise - noise arising from:

- **Production** of light in the phosphor
- **Transmission** of light through the optical system and
- **Transduction** of light into signal carriers in the optical image receptor

Both the light quanta and signal carriers are:

Secondary Quanta
Primary & Secondary Quantum Noise

Each stage involves a Random process.

The generation of X-ray quanta is governed by a Poisson process.

In general, we can treat the generation of light quantum from individual X-ray quanta as being Gaussian.

Stages 3 - 5 involve the selection of a fraction of the secondary quanta and thus are governed by Binomial processes.
Primary & Secondary Quantum Noise

The **Cascade** of these processes can be calculated mathematically.

However, a simple approach to estimating the dominant noise source in a medical image is to determine the number of quanta at each stage of the imaging cascade:

**the Stage with the Minimum Number of Quanta will be the Dominant Noise Source**
4.6 ANALYSIS OF SIGNAL & NOISE

4.6.1 Quantum Signal-to-Noise Ratio

There is a fundamental Difference between the high-contrast and low-contrast resolution of an imaging system.

In general, the high-contrast resolution is limited by the intrinsic blurring of the imaging system.

At some point, the system is unable to resolve two objects that are separated by a short distance, instead portraying them as a single object.

However, at Low Contrast, objects (even very large objects) may not be discernible because the signal of the object is substantially Lower than the noise in the region containing the object.
Generally, the Signal-to-Noise Ratio (SNR) is defined as the inverse of the Coefficient of Variation:

\[ SNR = \frac{\langle g \rangle}{\sigma_g} \]

where \( \langle g \rangle \) is the mean value \( \sigma_g \) the standard deviation

This definition of the SNR requires that a single pixel (or region) be measured repeatedly over various images (of the ensemble), provided that each measurement is independent (i.e. there is no correlation with time).
In an *Ergodic* system, the ensemble average can be replaced by an average over a region.

This definition is of value for photonic (or quantum) noise because in a uniform X-ray field, X-ray quanta are *Not* spatially correlated.

However, most imaging systems do blur the image to some degree, and hence introduce *Correlation* in the noise.

As a result, it is *Generally Inappropriate* to calculate pixel noise by analysing pixel values in a region for absolute noise calculations.
The **Definition** of SNR as

\[ SNR = \langle g \rangle / \sigma_g \]

is only useful when the image data are always positive, such as photon counts or luminance.

In systems where **Positivity** is not guaranteed, such as an ultrasound system, the SNR is defined as the **Power Ratio**, and is typically expressed in decibels:

\[ SNR_{dB} = 10 \log_{10} \frac{P_s}{P_n} = 10 \log_{10} \left( \frac{A_s}{A_n} \right)^2 = 20 \log_{10} \frac{A_s}{A_n} \]

where **P** is the average power and **A** is the root mean square amplitude of the signal, **s**, or noise, **n**.
Based on the work of Albert Rose, it is clear that the image quality of X-ray imaging systems is determined by the number of quanta used to produce an image.

This leads to the definition of the DQE:

**A Measure of the Fraction of the Quantum SNR of the Incident Quanta that is Recorded in the Image by an Imaging System**

Thus, the DQE is a measure of the **Fidelity** of an imaging system.
It is common to **Define** the DQE as:

\[
DQE = \frac{SNR_{out}^2}{SNR_{in}^2}
\]

where the \(SNR^2\) of the image is denoted by the subscript **out** and the \(SNR^2\) of the **Incident** X-ray quanta is given by:

\[
SNR_{in}^2 = \phi
\]

where \(\phi\) is the **Average** number of X-ray quanta incident on the detector.
Rodney Shaw introduced the concept of the DQE to medical imaging and also introduced the term **Noise-Equivalent Quanta** (NEQ)

The **NEQ** is the effective number of quanta needed to achieve a specific SNR in an ideal detector.

We can write:  

\[ DQE = \frac{NEQ}{\phi} \]

so that  

\[ NEQ = SNR_{out}^2 \]
In some sense,

the **NEQ** denotes the **Net Worth** of the image data in terms of X-ray quanta

and

the **DQE** defines the **Efficiency** with which an imaging system converts X-ray quanta into image data
This definition of DQE is the Zero Spatial Frequency value of the DQE.

Zero spatial frequency refers to a detector that counts X-ray quanta but does not produce a pixelated image; i.e. we only care about the Efficiency of counting the X-ray quanta.

Thus $\phi$ is a simple count of the X-ray quanta incident on the detector.
4.6 ANALYSIS OF SIGNAL & NOISE

4.6.2 Detective Quantum Efficiency

By this definition, an imaging system which:

- Perfectly absorbs each X-ray and
- Does not introduce any other noise

will **Perfectly Preserve** the SNR of the X-ray quanta

and hence:

\[
NEQ = \phi
\]

and

\[
DQE = 1
\]
If we consider an X-ray detector that is **Perfect** in every way **Except** that the quantum detection efficiency $\eta < 1.0$, we observe that while the incident number of quanta is again $\phi$, only $\eta \phi$ quanta are absorbed.

As a result:

\[
\text{NEQ} = \eta \phi \text{ and } \text{DQE} = \eta
\]

Thus, in this special instance, the DQE is equal to the quantum detection efficiency, $\eta$, the efficiency with which X-ray quanta are absorbed in the detector.
4.6 ANALYSIS OF SIGNAL & NOISE

4.6.2 Detective Quantum Efficiency

The **DQE** can be more generally expressed in terms of spatial frequencies:

\[
DQE(u, v) = \frac{SNR_{out}^2(u,v)}{SNR_{in}^2(u,v)}
\]

where we rely upon the property that X-ray quantum noise is **White**, leading to the \(SNR_{in}\) being a constant:

\[
SNR_{in}^2(u,v) = \Phi
\]

Here, \(\Phi\) is the **Photon Fluence** and has units of **Inverse Area**
4.6 ANALYSIS OF SIGNAL & NOISE
4.6.2 Detective Quantum Efficiency

The \( DQE(u, v) \) tells us how well the imaging system preserves the SNR_{in} at a specific spatial frequency, \((u,v)\)

In a similar fashion, \( NEQ(u, v) \) denotes the effective number of quanta that the image is worth at that frequency.
The NEQ and DQE can be calculated from Measurable Quantities - specifically:

\[
DQE(u, v) = \frac{\Phi \Gamma^2 T^2(u, v)}{W(u, v)} \quad \text{and} \quad NEQ(u, v) = \frac{\Phi^2 \Gamma^2 T^2(u, v)}{W(u, v)}
\]

It is clear from these equations that in an ideal system, the NPS is proportional to the MTF squared.

\[
W(u, v) = T^2(u, v)
\]
4.6   ANALYSIS OF SIGNAL & NOISE
4.6.2    Detective Quantum Efficiency

Standard measurement conditions for the NEQ and DQE have been specified by the IEC.

Most typically, an RQA-5 spectrum is used for radiography and RQA-M for mammography.

Tabulations of fluence as a function of air kerma are used in conjunction with measurements of kerma to calculate $\Phi$. 
4.6  ANALYSIS OF SIGNAL & NOISE
4.6.3  Signal-to-Noise Ratio

As we’ve defined it, the quantum SNR is related to the relative variation of pixel values in a Uniform Region

However, it is often necessary to compare the amplitude of a specific signal to the Background Noise

An Alternate Definition of the SNR is the difference in the means of two regions to the noise in those regions:

\[ SNR = \frac{|\langle x_a \rangle - \langle x_b \rangle|}{\sigma} \]

where \( x_a \) and \( x_b \) are the mean values in the region of an object (a) and background (b) and \( \sigma \) is the standard deviation of the background
A uniform disk (a) is shown on a uniform background (b) in the presence of X-ray quantum noise.

The SNR of the object is calculated as the difference in the mean signals divided by the noise characterized by the standard deviation ($\sigma$) of the Background.
4.6 ANALYSIS OF SIGNAL & NOISE

4.6.3 Signal-to-Noise Ratio

The choice of the background region is important:

the Standard Deviation should be Calculated using the Region that yields a Meaningful Result

For Example, if image processing (such as Thresholding) is used to force the background to a uniform value, then SNR as defined will be indefinite

Note that the SNR as defined goes by a number of names, including the signal difference to noise ratio (SdNR) and the contrast to noise ratio (CNR)
4.6 ANALYSIS OF SIGNAL & NOISE

4.6.3 Signal-to-Noise Ratio

The value of the SNR was first explained by Albert Rose who was interested in quantifying the quality of television images. Rose showed that an object is distinguishable from the background if the

\[ \text{SNR} \geq 5 \]

This can be related to a simple t-test in which an error rate of less than 1 in \(10^6\) occurs when the difference in the means is equal to 5 standard deviations.
Today, this is known as the **Rose Criterion** in imaging research.

It should be noted that the requirement of $\text{SNR} \geq 5$ is actually quite strict.

Depending upon the image task, it is possible to successfully operate at lower SNR values.
Assumption of the Rose model: the limiting factor in the detection of an object is the radiation dose (and hence number of X-ray quanta) used to produce the image.

This is True in an ideal imaging system.

In fact, the design of all imaging systems is driven by the goal of being X-ray (primary) quantum noise limited.
Robert Wagner has proposed a Taxonomy of noise limitations which is worth noting.

There are Four potential limitations in terms of the detection of objects:

1) Quantum Noise limited
2) Artefact limited
3) Anatomy limited
4) Observer limited
X Ray Quantum Noise Limited performance is the Preferred mode of operation, because the ability to detect or discriminate an object is determined solely by the radiation dose.

This is How All Detectors Should Operate

Ideally
Artefact limitation is the case in which the imaging system introduces artefacts which limit detection.

Classic examples include CT and MR where acquisition artefacts can predominate over the signal of interest.

Anatomic limited detection occurs when the normal anatomy (e.g. ribs in chest radiography or the breast parenchyma in mammography) mask the detection of objects, thereby reducing observer performance.
4.6 ANALYSIS OF SIGNAL & NOISE

4.6.3 Signal-to-Noise Ratio

Finally, there are situations in which the Observer is the limiting factor in performance.

For Example, a lesion may be readily visible, but the observer is distracted by an obvious benign or normal finding.

Thus Detection was Possible but did Not Occur.

In this chapter, we deal exclusively with quantum noise limited performance, which can be calculated using Ideal Observer methods.

In Chapter 18, the Modelling of Real Observers is discussed.
4.6 ANALYSIS OF SIGNAL & NOISE
4.6.3 Signal-to-Noise Ratio

Task-Specific

The MTF, NPS, NEQ and DQE are Frequency dependent characterizations of the detector.

However, these allow us to Calculate the image of a scene.

In particular, we can now use the SNR to quantify the ability of the detector to be used in:

- Signal Known Exactly (SKE),
- Background Known Exactly (BKE) tasks

assuming an ideal observer working with Gaussian statistics.
4.6 ANALYSIS OF SIGNAL & NOISE

4.6.3 Signal-to-Noise Ratio

Task-Specific

In this scenario, the observer is challenged with the task of deciding between Two hypotheses based upon a given set of data.

Under the First Hypothesis, the expected input signal is present, \( f_I \), and the image \( g_I \) is described by an appropriate Gaussian probability distribution.

Under Alternate Hypothesis, the expected input signal is absent, \( f_{II} \).
4.6 ANALYSIS OF SIGNAL & NOISE

4.6.3 Signal-to-Noise Ratio

The SNR of this task is given by:

\[ SNR_i^2 = \langle \Gamma^2 \iint |\Delta f(u,v)|^2 T(u,v)^2 \frac{1}{W(u,v)} du dv \rangle \]

where

\[ \Delta f(u, v) = f_I(u, v) - f_{II}(u, v) \]

is the difference between the signal Absent and Present
4.6 ANALYSIS OF SIGNAL & NOISE

4.6.3 Signal-to-Noise Ratio

Task-Specific

For a digital detector, $T$ is the presampled MTF, and thus we must account for aliasing by summing over all aliases of the signal:

$$SNR_I^2 = \langle \Gamma^2 \int\int \frac{\sum_{j,k} |\Delta f(u+u_j,v+v_k)|^2 T(u+u_j,v+v_k)^2}{W(u,v)} dudv \rangle$$

where the indices $j, k$ are used to index the aliases (in 2D)

In this way, we can calculate the SNR of the ideal observer for the detection of any object in an SKE/BKE task.
In Chapter 18, methods are described that extend this model to include characteristics of **Real Observers**
The **Ultimate Goal** of radiation safety in medical imaging is to obtain a **Figure of Merit** based on the maximum benefit to the patient for the smallest detriment.

We can now calculate the SNR for the detection of a known object (for example a tumour) on a known background.

This calculation is based upon parameters of a specific detector so that detectors can be ** Compared** or **Optimized**.
This calculation can act as a useful surrogate of the benefit, since a disease once detected can be treated.

We need therefore to relate this benefit to some **Metric of Risk**

A **Useful** metric:

\[
\frac{SNR^2}{E}
\]

where \( E \) is the **Effective Dose**
This metric is formulated with the SNR$^2$ based on the fact that in quantum noise limited imaging the

\[ SNR \propto \sqrt{\phi} \]

Thus the ratio is **Invariant** with dose

**Other Descriptors** of patient dose may also be useful, for example, optimization in terms of skin dose

Using this formation, it is possible, for example, to determine the **Optimal Radiographic Technique** (tube voltage, filtration, etc) for a specific task


INTERNATIONAL COMMISSION ON RADIATION UNITS AND MEASUREMENTS, Medical Imaging - The Assessment of Image Quality, ICRU Rep. 54 Bethesda, MD (1996)
